A new constrained mKP hierarchy and the generalized Darboux transformation for the mKP equation with self-consistent sources

Ting Xiao Yunbo Zeng†

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

†*Email: yzeng@math.tsinghua.edu.cn*

Abstract

The mKP equation with self-consistent sources (mKPESCS) is treated in the framework of the constrained mKP hierarchy. We introduce a new constrained mKP hierarchy which may be viewed as the stationary hierarchy of the mKP hierarchy with self-consistent sources. This offers a natural way to obtain the Lax representation for the mKPESCS. Based on the conjugate Lax pairs, we construct the generalized Darboux transformation with arbitrary functions in time t for the mKPESCS which, in contrast with the Darboux transformation for the mKP equation, provides a non-auto-Bäcklund transformation between two mKPESCSs with different degrees. The formula for n-times repeated generalized Darboux transformation is proposed and enables us to find the rational solutions (including the lump solutions), soliton solutions and the solutions of breather type of the mKPESCS.

Keywords: Lax representation; constrained mKP hierarchy; mKP equation with self-consistent sources(mKPESCS); Darboux transformation(DT); rational solution; soliton solution; solution of breather type

1 Introduction

Soliton equations with self-consistent sources (SESCSs) are important models in many fields of physics, such as hydrodynamics, solid state physics, plasma physics, etc. [1-8,15].

For example, the nonlinear Schrödinger equation with self-consistent sources represents the nonlinear interaction of an electrostatic high-frequency wave with the ion acoustic wave in a two component homogeneous plasma[8]. The KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves[4]. The KP equation with self-consistent sources describes the interaction of a long wave with a short-wave packet propagating on the x,y plane at an angle to each other(see [15] and the references therein). Until now, much development has been made in the study of SESCS. For example, in (1+1)-dimensional case, some SESCSs such as the KdV, modified KdV, nonlinear Schrödinger, AKNS and Kaup-Newell hierarchies with self-consistent sources were solved by the inverse scattering method [1,2,3,6-10]. Also a type of generalized binary Darboux transformations with arbitrary functions in time t for some (1+1)-dimensional SESCSs , which offer a non-auto-Bäcklund transformation between two SESCSs with different degrees of sources, have been constructed and can be used to obtain N-soliton, positon and negaton solution [12-14]. In (2+1)-dimensional case, some results to the SESCSs have been obtained. The soliton solution of the KP equation with self-consistent sources (KPESCS) was first found by Mel'nikov [15, 16]. However, since the explicit time part of the Lax representation of the KPESCS was not found, the method to solve the KPESCS by inverse scattering transformation in [15, 16] was quite complicated. In [17], in the framework of the constrained KP hierarchy, we get the Lax representation of the KP equation with self-consistent sources naturally and construct the generalized binary Darboux transformation for it naturally. The KPESCS is also studied by Hirota method in [18].

In this paper, we develop the idea presented in [17] to study the mKP equation with self-consistent sources. First we give a new constrained mKP hierarchy which may be viewed as the stationary hierarchy of the mKP hierarchy with self-consistent sources. This gives a natural way to find the Lax representation for the mKPESCS. Using the conjugate Lax pairs, we construct the generalized Darboux transformation with arbitrary functions in time t for the mKPESCS. In contrast with the Darboux transformation for the mKP equation which offers a Bäcklund transformation, this transformation provides a non-auto-Bäcklund transformation between two mKPESCSs with different degrees of sources. By this generalized Darboux transformation, some interesting solutions of mKPESCS such as soliton solutions, rational solutions (including lump solutions) and solutions of breather type are obtained.

The paper will be organized as follows. We recall some facts about the mKP hierarchy and mKP equation through the pseudo-differential operator (PDO) formalism in the next section. In section 3, we introduce a new constrained mKP hierarchy and give some examples of equations. In section 4, we reveal the relation between the mKP hierarchy with self-consistent

sources and the constrained mKP hierarchy given in the previous section. Then the conjugate Lax pairs of the mKP hierarchy with self-consistent sources can be obtained naturally. Using the conjugate Lax pairs, we can construct the generalized Darboux transformations with arbitrary functions in time for the mKPESCS. In Section 5, the n-times repeated generalized Darboux transformation will be constructed by which some interesting solutions for the mKPESCS are obtained in section 6.

2 The mKP hierarchy and the mKP equation

Let us consider the following pseudo-differential operator(PDO)

$$L = L_{mKP} = \partial + V + V_1\partial^{-1} + V_2\partial^{-2} + ..., \tag{2.1}$$

where ∂ denotes $\frac{\partial}{\partial x}$, and $V, V_j, j = 1, ...$ are functions. Denote $B_m = (L^m)_{\geq 1}$ for $\forall m \in N$ where $(L^m)_{\geq 1}$ represents the projection of L^m to its differential part whose order is more than 1. Then the mKP hierarchy is defined as [19]

$$L_{t_k} = [B_k, L], k \geq 1. \tag{2.2}$$

or the equivalent form

$$(L^n)_{t_k} = [B_k, L^n], n, k \geq 1. \tag{2.3}$$

The mKP hierarchy (2.2) can also be written in the zero-curvature form

$$(B_n)_{t_m} - (B_m)_{t_n} + [B_n, B_m] = 0, \quad n, m \geq 2. \tag{2.4}$$

The equation (2.4) has a pair of conjugate Lax pairs as follows

$$\psi_{1,t_m} = (B_m\psi_1), \tag{2.5a}$$

$$\psi_{1,t_n} = (B_n\psi_1), \tag{2.5b}$$

and

$$\psi_{2,t_m} = (\tilde{B}_m\psi_2), \tag{2.6a}$$

$$\psi_{2,t_n} = (\tilde{B}_n\psi_2), \tag{2.6b}$$

where $\tilde{B}_k = -(\partial B_k\partial^{-1})^*$, $k \geq 2$. We make a convention that for any operator P and a function f, (Pf) means that the operator P acts on f while Pf means the product of P

and f. It is easy to see that $\tilde{B}_k$ are also differential operators. When $n=2,\ m=3$ we get the mKP equation as follows

$$4V_{t_3} - V_{xxx} + 6V^2V_x - 3(D^{-1}V_{t_2t_2}) - 6V_x(D^{-1}V_{t_2}) = 0 \tag{2.7}$$

where $DD^{-1} = D^{-1}D = 1$. Set

$$u = -V, \quad t = -\frac{1}{4}t_3, \quad y = \alpha t_2, \tag{2.8}$$

the mKP equation will be written as

$$u_t - 6u^2u_x + u_{xxx} + 3\alpha^2(D^{-1}u_{yy}) - 6\alpha u_x(D^{-1}u_y) = 0, \tag{2.9}$$

which is called the mKPI equation when $\alpha = i$ and mKPII equation when $\alpha = 1$. From (2.5) and (2.6), we will get the conjugate Lax pairs of (2.9) respectively as follows

$$\alpha\psi_{1,y} = \psi_{1,xx} - 2u\psi_{1,x}, \tag{2.10a}$$

$$\psi_{1,t} = (A_1(u)\psi_1), \quad A_1(u) = -4\partial^3 + 12u\partial^2 - 6(-u_x + u^2 - \alpha D^{-1}u_y)\partial, \tag{2.10b}$$

and

$$\alpha\psi_{2,y} = -\psi_{2,xx} - 2u\psi_{2,x}, \tag{2.11a}$$

$$\psi_{2,t} = (A_2(u)\psi_2), \quad A_2(u) = -4\partial^3 - 12u\partial^2 - 6(u_x + u^2 - \alpha D^{-1}u_y)\partial, \tag{2.11b}$$

It is known that the system (2.10) is covariant w.r.t. the following transformations [20]

$$\psi_1[1] = \psi_1 - f_1\frac{\int \psi_{1,x}g_1\mathrm{d}x + C_2}{\int f_{1,x}g_1\mathrm{d}x - C_1}, \tag{2.12a}$$

$$u[1] = u + \partial_x\mathrm{ln}\frac{\int g_{1,x}f_1\mathrm{d}x + C_1}{\int f_{1,x}g_1\mathrm{d}x - C_1}, \tag{2.12b}$$

while the system (2.11) is covariant w.r.t.

$$\psi_2[1] = \psi_2 - g_1\frac{\int \psi_{2,x}f_1\mathrm{d}x + C_2}{\int g_{1,x}f_1\mathrm{d}x + C_1}, \tag{2.13a}$$

$$u[1] = u + \partial_x\mathrm{ln}\frac{\int g_{1,x}f_1\mathrm{d}x + C_1}{\int f_{1,x}g_1\mathrm{d}x - C_1}, \tag{2.13b}$$

where f_1, g_1 are solutions of (2.10) and (2.11) respectively and C_1, C_2 are arbitrary constants. We point out that throughout the paper, the integral operation $\int f_1f_2\mathrm{d}x$ means $\int_{-\infty}^{x} f_1f_2\mathrm{d}x$ or $-\int_x^{\infty} f_1f_2\mathrm{d}x$ and contains no arbitrary function of y and t, only numerical constant if we

impose some suitable boundary condition on the integrand functions f_1 and f_2 at $x=-\infty$ or $x=\infty$. Substituting (2.12a) (with $C_2=0$,$C_1=C$) into (2.10b), we will get the following identity

$$\begin{aligned}
& (A_1(u[1])\psi_1[1]) \\
= & (\psi_1 - f_1\tfrac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C})_t \\
= & \psi_{1,t} - f_{1,t}\tfrac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C} - f_1\tfrac{\int(\psi_{1,x}g_{1,t}+\psi_{1,xt}g_1)\mathrm{d}x(\int f_{1,x}g_1\mathrm{d}x-C)-(\int\psi_{1,x}g_1\mathrm{d}x)[\int(g_{1,t}f_{1,x}+g_1f_{1,xt})\mathrm{d}x]}{(\int f_{1,x}g_1\mathrm{d}x-C)^2} \\
= & (A_1(u)\psi_1) - (A_1(u)f_1)\tfrac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C} \\
& -f_1\tfrac{[\int(\psi_{1,x}(A_2(u)g_1)+(A_1(u)\psi_1)_xg_1)\mathrm{d}x](\int f_{1,x}g_1\mathrm{d}x-C)-(\int\psi_{1,x}g_1\mathrm{d}x)[\int((A_2(u)g_1)f_{1,x}+g_1(A_1(u)f_1)_x)\mathrm{d}x]}{(\int f_{1,x}g_1\mathrm{d}x-C)^2}.
\end{aligned} \tag{2.14}$$

3 A new constraint of the mKP hierarchy

In [19], W.Oevel and W.Strampp have studied the constraint of the PDO L (2.1) as

$$L^n = (L^n)_{\geq 1} + v_0 + \partial^{-1}\psi, \tag{3.1}$$

from which we will get the Kaup-Broer hierarchy when $n=1$. Here we consider a new constraint as follows

$$L^n = (L^n)_{\geq 1} + q\partial^{-1}r\partial. \tag{3.2}$$

where q,r satisfy that

$$q_{t_k} = (B_kq), \quad r_{t_k} = (\tilde{B}_kr), \tag{3.3}$$

and $B_k = ((L^n)^{\frac{k}{n}})_{\geq 1} = [((L^n)_{\geq 1} + q\partial^{-1}r\partial)^{\frac{k}{n}}]_{\geq 1}$.
Then a new n-constrained mKP hierarchy will be obtained as

$$(L^n)_{t_k} = [(L^k)_{\geq 1}, L^n] = [B_k, L^n], \tag{3.4a}$$

$$q_{t_k} = (B_kq), \tag{3.4b}$$

$$r_{t_k} = (\tilde{B}_kr), \tag{3.4c}$$

First, we will prove that the constraint (3.2) together with the condition (3.3) is compatible with the mKP hierarchy (2.2). The following formulas for PDO will be useful in the proof and we list them below,

$$(\Lambda^*)_0 = res(\partial^{-1}\Lambda), \quad (\Lambda)_0 = res(\Lambda\partial^{-1}), \quad (\Lambda\partial^{-1})_{<0} = (\Lambda)_0\partial^{-1} + (\Lambda)_{<0}\partial^{-1}, \tag{3.5a}$$

$$(Pq\partial^{-1}r)_{<0} = (Pq)\partial^{-1}r, \quad (q\partial^{-1}rP)_{<0} = q\partial^{-1}(P^*r), \tag{3.5b}$$

where Λ is an arbitrary PDO and P differential operator. $(A)_0$ denote the zero order term for a PDO A.

Theorem 3.1 *The constraint (3.2) together with the condition (3.3) is compatible with the mKP hierarchy (2.2).*

Proof: We need to prove the following identity

$$(q\partial^{-1}r\partial)_{t_k} = [B_k, L^n]_{\le 0} = [B_k, q\partial^{-1}r\partial]_{\le 0}, \tag{3.6}$$

$$\begin{aligned} \text{the l.h.s. of(3.6)} &= q_{t_k}\partial^{-1}r\partial + q\partial^{-1}r_{t_k}\partial \\ &= (B_kq)\partial^{-1}r\partial + q\partial^{-1}(\tilde{B}_kr)\partial \\ &\triangleq l_1 + l_2 \end{aligned} \tag{3.7}$$

$$\begin{aligned} \text{the r.h.s. of(3.6)} &= (B_kq\partial^{-1}r\partial)_{\le 0} - (q\partial^{-1}r\partial B_k)_{\le 0} \\ &\triangleq r_1 - r_2 \end{aligned} \tag{3.8}$$

$$(l_1)_0 = ((B_kq)\partial^{-1}r\partial)_0 = (B_kq)r, \quad (l_2)_0 = (q\partial^{-1}(\tilde{B}_kr)\partial)_0 = q(\tilde{B}_kr), \tag{3.9}$$

$$\begin{aligned} (r_1)_0 &= res[\partial^{-1}(r_1)^*] = res[\partial^{-1}(\partial r\partial^{-1}qB_k^*)] = res(r\partial^{-1}qB_k^*) \\ &= res(r\partial^{-1}qB_k^*)_{<0} = res(r\partial^{-1}(B_kq)) = r(B_kq), \end{aligned} \tag{3.10}$$

$$\begin{aligned} (r_2)_0 &= (q\partial^{-1}r\partial B_k)_0 = res[q\partial^{-1}r\partial B_k\partial^{-1}] = res[q\partial^{-1}r(\partial B_k\partial^{-1})] \\ &= res[q\partial^{-1}((\partial B_k\partial^{-1})^*r)] = q((\partial B_k\partial^{-1})^*r) = -q(\tilde{B}_kr), \end{aligned} \tag{3.11}$$

So

$$(l_1)_0 + (l_2)_0 = (r_1)_0 - (r_2)_0. \tag{3.12}$$

$$(l_1)_{<0} = ((B_kq)\partial^{-1}r\partial)_{<0} = -(B_kq)\partial^{-1}r_x, \tag{3.13}$$

$$(l_2)_{<0} = (q\partial^{-1}(\tilde{B}_kr)\partial)_{<0} = -q\partial^{-1}[\partial((\tilde{B}_kr))] = q\partial^{-1}[\partial\partial^{-1}(B_k^*\partial r)] = q\partial^{-1}(B_k^*r_x), \tag{3.14}$$

By the last formula of (3.5a), we have

$$(r_1\partial^{-1})_{<0} = (B_kq\partial^{-1}r)_{<0} = (r_1)_0\partial^{-1} + (r_1)_{<0}\partial^{-1}, \tag{3.15}$$

i.e.

$$(B_kq)\partial^{-1}r = (r_1)_0\partial^{-1} + (r_1)_{<0}\partial^{-1}.$$

Multiplying ∂ on the right and taking the negative part of both sides of the above identity, we get

$$((B_kq)\partial^{-1}r\partial)_{<0} = (r_1)_{<0}$$

So

$$(r_1)_{<0} = -(B_kq)\partial^{-1}r_x.$$

$$(r_2)_{<0} = (q\partial^{-1}r\partial B_k)_{<0} = q\partial^{-1}(\partial B_k)^*(r) = -q\partial^{-1}(B_k^*r_x) \tag{3.16}$$

So we have

$$(l_1)_{<0} + (l_2)_{<0} = (r_1)_{<0} - (r_2)_{<0} \tag{3.17}$$

From (3.12) and (3.17), we can see (3.6) holds.
This completes the proof.

We give some examples below.
(a) 1-constraint($n = 1$).
Here

$$L = \partial + q\partial^{-1}r\partial. \tag{3.18}$$

So

$$V = qr, \quad V_1 = -qr_x, \quad ... \tag{3.19}$$

$$B_2 = (L^2)_{\geq 1} = \partial^2 + 2qr\partial, \qquad \tilde{B}_2 = -(\partial B_2 \partial^{-1})^* = -\partial^2 + 2qr\partial,$$

$$B_3 = (L^3)_{\geq 1} = \partial^3 + 3qr\partial^2 + (3q^2r^2 + 3q_xr)\partial,$$

$$\tilde{B}_3 = -(\partial B_3 \partial^{-1})^* = \partial^3 - 3qr\partial^2 + (3q^2r^2 - 3qr_x)\partial, \quad ...$$

The first two equations of the 1-constrained hierarchy are

$$q_{t_2} = q_{xx} + 2qrq_x, \tag{3.20a}$$

$$r_{t_2} = -r_{xx} + 2qrr_x, \tag{3.20b}$$

and

$$q_{t_3} = q_{xxx} + 3qrq_{xx} + (3q^2r^2 + 3q_xr)q_x, \tag{3.21a}$$

$$r_{t_3} = r_{xxx} - 3qrr_{xx} + (3q^2r^2 - 3qr_x)r_x. \tag{3.21b}$$

Equation (3.20) is the generalized NS equation with derivative coupling given by Chen et al [21, 22]. The constrained hierarchy is also studied in [23].
(b) 2-constraint($n = 2$).
Here

$$L^2 = \partial^2 + 2V\partial + q\partial^{-1}r\partial. \tag{3.22}$$

from which we find

$$V_1 = qr - V_x - V^2, \qquad ...$$

$$B_2 = (L^2)_{\geq 1} = \partial^2 + 2V\partial, \qquad \tilde{B}_2 = -(\partial B_2 \partial^{-1})^* = -\partial^2 + 2V\partial,$$

$$B_3 = (L^3)_{\geq 1} = \partial^3 + 3V\partial^2 + 3qr\partial, \qquad \tilde{B}_3 = -(\partial B_3 \partial^{-1})^* = \partial^3 - 3V\partial^2 + (3qr - 3V_x)\partial, \quad ...$$

The first two equations of the 2-constrained hierarchy are

$$V_{t_2} = (qr)_x, \tag{3.23a}$$

$$q_{t_2} = q_{xx} + 2Vq_x, \tag{3.23b}$$

$$r_{t_2} = -r_{xx} + 2Vr_x. \tag{3.23c}$$

and

$$V_{t_3} = V_{xxx} + 3VV_{xx} + 6V^2V_x + 3qrV_x - \frac{3}{2}(qr_x)_x, \tag{3.24a}$$

$$q_{t_3} = q_{xxx} + 3Vq_{xx} + 3qrq_x, \tag{3.24b}$$

$$r_{t_3} = r_{xxx} - 3Vr_{xx} + (3qr - 3V_x)r_x. \tag{3.24c}$$

(c) 3-constraint($n = 3$).
Here

$$L^3 = \partial^3 + 3V\partial^2 + 3(V^2 + V_x + V_1)\partial + q\partial^{-1}r\partial. \tag{3.25}$$

The first equation of the 3-constrained hierarchy is

$$V_{t_2} = V_{xx} + 2V_{1,x} + 2VV_x, \tag{3.26a}$$

$$3V_{1,t_2} = -2V_{xxx} - 6VV_{xx} - 6V^2V_x - 6V_x^2 - 3V_{1,xx} - 6VV_{1,x} - 6V_1V_x + 2(qr)_x, \tag{3.26b}$$

$$q_{t_2} = q_{xx} + 2Vq_x, \tag{3.26c}$$

$$r_{t_2} = -r_{xx} + 2Vr_x. \tag{3.26d}$$

Eliminating V_1 from the above equation, we get

$$\frac{1}{2}V_{xxx} + \frac{3}{2}D^{-1}(V_{yy}) + 3(D^{-1}V_y)V_x - 3V^2V_x - 2(qr)_x = 0, \tag{3.27a}$$

$$q_{t_2} = q_{xx} + 2Vq_x, \tag{3.27b}$$

$$r_{t_2} = -r_{xx} + 2Vr_x. \tag{3.27c}$$

4 The mKP equation with self-consistent sources and its generalized Darboux transformation

If generalizing the constraint (3.2) to

$$L^n = (L^n)_{\geq 1} + \sum_{i=1}^{N} q_i\partial^{-1}r_i\partial. \tag{4.1}$$

where

$$q_{i,t_k} = (B_k q_i), \quad r_{i,t_k} = (\tilde{B}_k r_i), \tag{4.2}$$

and adding the term $(B_k)_{t_n}$ to the right hand side of (3.4a), we can define the mKP hierarchy with self-consistent sources as follows

$$(B_k)_{t_n} - (L^n)_{t_k} + [B_k, L^n] = 0, \tag{4.3a}$$

$$q_{i,t_k} = (B_k q_i), \tag{4.3b}$$

$$r_{i,t_k} = (\tilde{B}_k r_i). \tag{4.3c}$$

So if the variable "t_n" is viewed as the evolution variable, the n-constrained mKP hierarchy may be regarded as the stationary hierarchy of the mKP hierarchy with self-consistent sources. Under the condition (4.3b) and (4.3c), we naturally get the conjugate Lax pairs of (4.3a) as follows

$$\psi_{1,t_k} = (B_k \psi_1), \tag{4.4a}$$

$$\psi_{1,t_n} = (L^n \psi_1) = (B_n \psi_1) + \sum_{i=1}^{N} q_i \int r_i \psi_{1,x} \mathrm{d}x, \tag{4.4b}$$

and

$$\psi_{2,t_k} = (\tilde{B}_k \psi_2), \tag{4.5a}$$

$$\psi_{2,t_n} = (\tilde{L}^n \psi_2) = (\tilde{B}_n \psi_2) - ([\partial(\sum_{i=1}^{N} q_i \partial^{-1} r_i \partial)\partial^{-1}]^* \psi_2) = (\tilde{B}_n \psi_2) - \sum_{i=1}^{N} r_i \int q_i \psi_{2,x} \mathrm{d}x, \tag{4.5b}$$

When $n = 3$, $k = 2$, under the transformation (2.8) and setting

$$\Phi_i = r_i, \qquad \Psi_i = q_i,$$

we will get the mKP equation with self-consistent sources (mKPESCS) and its conjugate Lax pairs respectively from (4.3),(4.4) and (4.5).
The mKPESCS is

$$u_t + u_{xxx} + 3\alpha^2 D^{-1}(u_{yy}) - 6\alpha D^{-1}(u_y)u_x - 6u^2 u_x + 4\sum_{i=1}^{N}(\Psi_i \Phi_i)_x = 0, \tag{4.6a}$$

$$\alpha \Psi_{i,y} = \Psi_{i,xx} - 2u\Psi_{i,x}, \tag{4.6b}$$

$$\alpha \Phi_{i,y} = -\Phi_{i,xx} - 2u\Phi_{i,x}, \tag{4.6c}$$

which is called the mKPIESCS when $\alpha = i$ and mKPIIESCS when $\alpha = 1$. Under the condition (4.6b) and (4.6c), the conjugate Lax pairs for (4.6a) are

$$\alpha\psi_{1,y} = \psi_{1,xx} - 2u\psi_{1,x}, \tag{4.7a}$$

$$\psi_{1,t} = (A_1(u)\psi_1) + T_N^1(\Psi,\Phi)\psi_1, \quad T_N^1(\Psi,\Phi)\psi_1 = -4\sum_{i=1}^{N}\Psi_i\int\Phi_i\psi_{1,x}\mathrm{d}x, \tag{4.7b}$$

and

$$\alpha\psi_{2,y} = -\psi_{2,xx} - 2u\psi_{2,x}, \tag{4.8a}$$

$$\psi_{2,t} = (A_2(u)\psi_2) + T_N^2(\Psi,\Phi)\psi_2, \quad T_N^2(\Psi,\Phi)\psi_2 = 4\sum_{i=1}^{N}\Phi_i\int\Psi_i\psi_{2,x}\mathrm{d}x, \tag{4.8b}$$

For the system (4.7), we can construct the following Darboux transformation.

Theorem 4.1 *Assume $u, \Phi_1, ..., \Phi_N, \Psi_1, ..., \Psi_N$ be a solution of the mKPESCS (4.6) and f_1, g_1 satisfy (4.7) and (4.8) respectively, then the system (4.7) has the following Darboux transformation*

$$\psi_1[1] = \psi_1 - f_1\frac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x - C}, \tag{4.9a}$$

$$u[1] = u + \partial_x\ln\frac{\int g_{1,x}f_1\mathrm{d}x + C}{\int f_{1,x}g_1\mathrm{d}x - C} = u + \frac{g_{1,x}f_1}{\int g_{1,x}f_1\mathrm{d}x + C} - \frac{f_{1,x}g_1}{\int f_{1,x}g_1\mathrm{d}x - C}, \tag{4.9b}$$

$$\Psi_j[1] = \Psi_j - f_1\frac{\int\Psi_{i,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x - C}, \tag{4.9c}$$

$$\Phi_j[1] = \Phi_j - g_1\frac{\int f_1\Phi_{j,x}\mathrm{d}x}{\int f_1g_{1,x}\mathrm{d}x + C}, \qquad j = 1, ..., N. \tag{4.9d}$$

where C is an arbitrary constant.

Proof: It is obvious that $u[1]$, $\psi_1[1]$, $\Phi_i[1]$,$\Psi_i[1]$, $i = 1, ..., N$ satisfy (4.6b),(4.6c) and (4.7a). So we only need to prove that (4.7b) holds, i.e., to prove the following equality

$$\psi_1[1]_t = (A_1(u[1])\psi_1[1]) + T_N^1(\Psi[1],\Phi[1])\psi_1[1]. \tag{4.10}$$

Using (4.7b), we have

$$\begin{aligned}\psi_1[1]_t \;=\;& (A_1(u)\psi_1) + T_N^1(\Psi,\Phi)\psi_1 - ((A_1(u)f_1) + T_N^1(\Psi,\Phi)f_1)\tfrac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x - C}\\ &-f_1\tfrac{\int((A_2(u)g_1)+T_N^2(\Psi,\Phi)g_1)\psi_{1,x}\mathrm{d}x+\int g_1((A_1(u)\psi_1)+T_N^1(\Psi,\Phi)\psi_1)_x\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C}\\ &+f_1(\textstyle\int g_1\psi_{1,x}\mathrm{d}x)\tfrac{\int((A_2(u)g_1)+T_N^2(\Psi,\Phi)g_1)f_{1,x}\mathrm{d}x+\int g_1((A_1(u)f_1)+T_N^1(\Psi,\Phi)f_1)_x\mathrm{d}x}{(\int f_{1,x}g_1\mathrm{d}x-C)^2}\end{aligned} \tag{4.11}$$

It is easy to verify that (2.14) still holds now. So we only need to prove the following identity

$$\begin{array}{rl} & T_N^1(\Psi[1],\Phi[1])\psi_1[1] \\ = & T_N^1(\Psi,\Phi)\psi_1 - T_N^1(\Psi,\Phi)f_1\frac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C} - f_1\frac{\int T_N^2(\Psi,\Phi)g_1\psi_{1,x}\mathrm{d}x+\int g_1(T_N^1(\Psi,\Phi)\psi_1)_x\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C} \\ & +f_1(\int g_1\psi_{1,x}\mathrm{d}x)\frac{\int T_N^2(\Psi,\Phi)g_1f_{1,x}\mathrm{d}x+\int g_1(T_N^1(\Psi,\Phi)f_1)_x\mathrm{d}x}{(\int f_{1,x}g_1\mathrm{d}x-C)^2} \end{array} \tag{4.12}$$

By substituting the expression of T_N^1 and T_N^2 in (4.7b) and (4.8b), we find

$$\begin{array}{rl} & \text{the r.h.s. of (4.12)} \\ = & -4\sum_{j=1}^N\Psi_j\int\Phi_j\psi_{1,x}\mathrm{d}x + 4f_1\sum_{j=1}^N\frac{(\int g_1\Psi_{j,x}\mathrm{d}x)(\int\Phi_j\psi_{1,x}\mathrm{d}x)}{\int f_{1,x}g_1\mathrm{d}x-C} \\ & +4\sum_{j=1}^N(\Psi_j - f_1\frac{\int g_1\Psi_{j,x}\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C})\frac{(\int g_1\psi_{1,x}\mathrm{d}x)(\int\Phi_jf_{1,x}\mathrm{d}x)}{\int f_{1,x}g_1\mathrm{d}x-C}. \end{array} \tag{4.13}$$

Then using (4.9) and (4.7b), we can show that

$$\begin{array}{rl} & \text{the l.h.s. of (4.12)} \\ = & -4\sum_{j=1}^N\Psi_j\int\Phi_j\psi_{1,x}\mathrm{d}x + 4f_1\sum_{j=1}^N\frac{(\int g_1\Psi_{j,x}\mathrm{d}x)(\int\Phi_j\psi_{1,x}\mathrm{d}x)}{\int f_{1,x}g_1\mathrm{d}x-C} \\ & +4\sum_{j=1}^N(\Psi_j - f_1\frac{\int g_1\Psi_{j,x}\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C})\frac{(\int g_1\psi_{1,x}\mathrm{d}x)(\int\Phi_jf_{1,x}\mathrm{d}x)}{\int f_{1,x}g_1\mathrm{d}x-C} \\ = & \text{the r.h.s. of (4.12).} \end{array} \tag{4.14}$$

This completes the proof.

If C is replaced by $C(t)$, an arbitrary function in time t in (4.9), then (4.6b),(4.6c) and (4.7a) are also covariant w.r.t. (4.9), but (4.7b) is not covariant w.r.t. (4.9) any longer. In fact, we have the following theorem.

Theorem 4.2 *Given $u,\Psi_1,...,\Psi_N,\Phi_1,...,\Phi_N$ a solution of the mKPESCS (4.6) and let f_1 and g_1 be solutions of the system (4.7) and (4.8) respectively, then the transformation with $C(t)$ (an arbitrary function in t) defined by*

$$\psi_1[1] = \psi_1 - f_1\frac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x - C(t)}, \tag{4.15a}$$

$$u[1] = u + \partial_x\ln\frac{\int g_{1,x}f_1\mathrm{d}x + C(t)}{\int f_{1,x}g_1\mathrm{d}x - C(t)} = u + \frac{g_{1,x}f_1}{\int g_{1,x}f_1\mathrm{d}x + C(t)} - \frac{f_{1,x}g_1}{\int f_{1,x}g_1\mathrm{d}x - C(t)}, \tag{4.15b}$$

$$\Psi_j[1] = \Psi_j - f_1\frac{\int\Psi_{i,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x - C(t)}, \tag{4.15c}$$

$$\Phi_j[1] = \Phi_j - g_1\frac{\int f_1\Phi_{j,x}\mathrm{d}x}{\int f_1g_{1,x}\mathrm{d}x + C(t)}, \qquad j = 1,...,N, \tag{4.15d}$$

and

$$\Psi_{N+1}[1]=-\frac{1}{2}\frac{\sqrt{\dot{C}(t)}f_1}{\int f_{1,x}g_1\mathrm{d}x-C(t)},\quad \Phi_{N+1}[1]=\frac{1}{2}\frac{\sqrt{\dot{C}(t)}g_1}{\int g_{1,x}f_1\mathrm{d}x+C(t)},\tag{4.15e}$$

transforms (4.6b),(4.6c) and (4.7) respectively into

$$\alpha\Psi_i[1]_y=\Psi_i[1]_{xx}-2u[1]\Psi_i[1]_x,\tag{4.16a}$$

$$\alpha\Phi_i[1]_y=-\Phi_i[1]_{xx}-2u[1]\Phi_i[1]_x,\quad i=1,...,N+1,\tag{4.16b}$$

$$\alpha\psi_1[1]_y=\psi_1[1]_{xx}-2u[1]\psi_1[1]_x,\tag{4.16c}$$

$$\psi_1[1]_t=A_1(u[1])\psi_1[1]+T^1_{N+1}(\Psi[1],\Phi[1])\psi_1[1].\tag{4.16d}$$

So $u[1]$, $\Psi_i[1]$, $\Phi_i[1]$, $i=1,...,N+1$ *is a new solution of the mKPESCS (4.6) with degree* $N+1$.

Proof: Equations (4.16a), (4.16b) and (4.16c) hold obviously. We only need to prove (4.16d). Substituting (4.15a) into the left hand side of (4.16d) and using the result of the previous theorem, we have

$$\psi_1[1]_t=(\psi_1-f_1\frac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C(t)})_t=A_1(u[1])\psi_1[1]+T^1_N(\Psi[1],\Phi[1])\psi_1[1]-\frac{\dot{C}(t)f_1\int\psi_{1,x}g_1\mathrm{d}x}{(\int f_{1,x}g_1\mathrm{d}x-C(t))^2},\tag{4.17}$$

So we only need to prove

$$-4\Psi_{N+1}[1]\int\Phi_{N+1}[1]\psi_{1,x}[1]\mathrm{d}x=-\frac{\dot{C}(t)f_1\int\psi_{1,x}g_1\mathrm{d}x}{(\int f_{1,x}g_1\mathrm{d}x-C(t))^2},$$

i.e.

$$\frac{\dot{C}(t)f_1}{\int f_{1,x}g_1\mathrm{d}x-C(t)}\int\frac{g_1}{\int f_1g_{1,x}\mathrm{d}x+C(t)}(\psi_1-f_1\frac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C(t)})_x\mathrm{d}x=-\frac{\dot{C}(t)f_1\int\psi_{1,x}g_1\mathrm{d}x}{(\int f_{1,x}g_1\mathrm{d}x-C(t))^2},$$

i.e., to prove

$$\int\frac{g_1}{\int f_1g_{1,x}\mathrm{d}x+C(t)}(\psi_1-f_1\frac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C(t)})_x\mathrm{d}x=-\frac{\int\psi_{1,x}g_1\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C(t)},\tag{4.18}$$

$$\begin{aligned}
&\text{the l.h.s of (4.18)}\\
=\;&\int\tfrac{g_1}{\int f_1g_{1,x}\mathrm{d}x+C(t)}(\psi_{1,x}-\tfrac{f_{1,x}\int g_1\psi_{1,x}\mathrm{d}x+f_1g_1\psi_{1,x}}{\int f_{1,x}g_1\mathrm{d}x-C(t)}+f_1g_1f_{1,x}\tfrac{\int g_1\psi_{1,x}\mathrm{d}x}{(\int f_{1,x}g_1\mathrm{d}x-C(t))^2})\mathrm{d}x\\
=\;&\int[-\tfrac{g_1\psi_{1,x}}{\int f_{1,x}g_1\mathrm{d}x-C(t)}+\tfrac{g_1f_{1,x}\int g_1\psi_{1,x}\mathrm{d}x}{(\int f_{1,x}g_1\mathrm{d}x-C(t))^2}]\mathrm{d}x\\
=\;&-\int(\tfrac{\int g_1\psi_{1,x}\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C(t)})_x\mathrm{d}x\\
=\;&-\tfrac{\int g_1\psi_{1,x}\mathrm{d}x}{\int f_{1,x}g_1\mathrm{d}x-C(t)}\\
=\;&\text{the r.h.s of (4.18).}
\end{aligned}\tag{4.19}$$

This completes the proof.

Remark: If $C(t)$ is not a constant, i.e. $\frac{d}{dt}C(t) \neq 0$, the DT (4.15) provides a non-auto-Bäcklund transformation between two mKPESCSs (4.6) with degree N and $N+1$ respectively.

5 The n-times Repeated Generalized Darboux Transformation for the mKPESCS

Assuming f_1, ...,f_n are n arbitrary solutions of (4.7) and g_1, ...,g_n are n arbitrary solutions of (4.8), $C_1(t), ..., C_n(t)$ are n arbitrary functions in t, we define the following Wronskians:

$$\begin{array}{rcl} W_1(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t)) & = & det(X_{n\times n}), \\ W_2(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t)) & = & det(\tilde{X}_{n\times n}), \\ W_3(f_1,...,f_n;g_1,...,g_{n-1};C_1(t),...,C_{n-1}(t)) & = & det(Y_{n\times n}), \\ W_4(f_1,...,f_{n-1};g_1,...,g_n;C_1(t),...,C_{n-1}(t)) & = & det(\tilde{Y}_{n\times n}), \end{array} \tag{5.1}$$

where

$$X_{i,j} = -\delta_{i,j}C_i(t) + \int g_j f_{i,x}\mathrm{d}x, \tag{5.2a}$$

$$\tilde{X}_{i,j} = \delta_{i,j}C_i(t) + \int g_{j,x} f_i\mathrm{d}x, \quad i,j=1,...,n, \tag{5.2b}$$

$$Y_{i,j} = -\delta_{i,j}C_i(t) + \int g_i f_{j,x}\mathrm{d}x,\ i=1,...,n-1,\ j=1,...,n;\ \ Y_{n,j}=f_j,\ j=1,...,n. \tag{5.2c}$$

$$\tilde{Y}_{i,j} = \delta_{i,j}C_i(t) + \int g_{j,x} f_i\mathrm{d}x,\ i=1,...,n-1,\ j=1,...,n;\ \ \tilde{Y}_{n,j}=g_j,\ j=1,...,n. \tag{5.2d}$$

Lemma 5.1 *Assume $f_1,...,f_n$ are solutions of (4.7) and $g_1,...,g_n$ are solutions of (4.8), then for $2\leq m\leq n$, $1\leq k\leq n-m$, we have*

$$\begin{array}{rl} & W_1(f_m[m-1],...,f_{m+k}[m-1];g_m[m-1],...,g_{m+k}[m-1];C_m(t),...,C_{m+k}(t)) \\ = & \frac{W_1(f_{m-1}[m-2],...,f_{m+k}[m-2];g_{m-1}[m-2],...,g_{m+k}[m-2];C_{m-1}(t),...,C_{m+k}(t))}{-C_{m-1}(t)+\int f_{m-1}[m-2]_x g_{m-1}[m-2]\mathrm{d}x}, \end{array} \tag{5.3a}$$

$$\begin{array}{rl} & W_2(f_m[m-1],...,f_{m+k}[m-1];g_m[m-1],...,g_{m+k}[m-1];C_m(t),...,C_{m+k}(t)) \\ = & \frac{W_2(f_{m-1}[m-2],...,f_{m+k}[m-2];g_{m-1}[m-2],...,g_{m+k}[m-2];C_{m-1}(t),...,C_{m+k}(t))}{C_{m-1}(t)+\int f_{m-1}[m-2] g_{m-1}[m-2]_x\mathrm{d}x}, \end{array} \tag{5.3b}$$

$$\begin{array}{rl} & W_3(f_m[m-1],...,f_{m+k}[m-1];g_m[m-1],...,g_{m+k-1}[m-1];C_m(t),...,C_{m+k-1}(t)) \\ = & \frac{W_3(f_{m-1}[m-2],...,f_{m+k}[m-2];g_{m-1}[m-2],...,g_{m+k-1}[m-2];C_{m-1}(t),...,C_{m+k-1}(t))}{-C_{m-1}(t)+\int f_{m-1}[m-2]_x g_{m-1}[m-2]\mathrm{d}x}, \end{array} \tag{5.3c}$$

$$W_4(f_m[m-1],...,f_{m+k-1}[m-1];g_m[m-1],...,g_{m+k}[m-1];C_m(t),...,C_{m+k-1}(t))$$
$$= \frac{W_4(f_{m-1}[m-2],...,f_{m+k-1}[m-2];g_{m-1}[m-2],...,g_{m+k}[m-2];C_{m-1}(t),...,C_{m+k-1}(t))}{C_{m-1}(t)+\int f_{m-1}[m-2]g_{m-1}[m-2]_x \mathrm{d}x}. \tag{5.3d}$$

This lemma can be proved in the same way as we did in [17]. Then we have

Theorem 5.1 *Assume that* $u, \Psi_1, \cdots, \Psi_N, \Phi_1, \cdots, \Phi_N$ *is a solution of the mKPESCS (4.6),* $f_1, \cdots, f_n$ *and* $g_1, \cdots, g_n$ *are solutions of (4.7) and (4.8) respectively,* $C_1(t), ..., C_n(t)$ *are* n *arbitrary functions in* t*. Then the* n*-times repeated generalized Darboux transformation for (4.7) is given by*

$$\psi_1[n] = \frac{W_3(f_1,...,f_n,\psi_1;g_1,...,g_n;C_1(t),...,C_n(t))}{W_1(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t))}, \tag{5.4a}$$

$$u[n] = u + \partial_x \ln \frac{W_2(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t))}{W_1(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t))}, \tag{5.4b}$$

$$\Psi_i[n] = \frac{W_3(f_1,...,f_n,\Psi_i;g_1,...,g_n;C_1(t),...,C_n(t))}{W_1(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t))}, \tag{5.4c}$$

$$\Phi_i[n] = \frac{W_4(f_1,...,f_n;g_1,...,g_n,\Phi_i;C_1(t),...,C_n(t))}{W_2(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t))}, \tag{5.4d}$$

$$\Psi_{N+j}[n]$$
$$= -\frac{1}{2}\sqrt{\dot{C}_j(t)}\frac{W_3(f_1,...,f_{j-1},f_{j+1},...,f_n,f_j;g_1,...,g_{j-1},g_{j+1},...,g_n;C_1(t),...,C_{j-1}(t),C_{j+1}(t),...,C_n(t))}{W_1(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t))}, \tag{5.4e}$$

$$\Phi_{N+j}[n]$$
$$= \frac{1}{2}\sqrt{\dot{C}_j(t)}\frac{W_4(f_1,...,f_{j-1},f_{j+1},...,f_n;g_1,...,g_{j-1},g_{j+1},...,g_n,g_j;C_1(t),...,C_{j-1}(t),C_{j+1}(t),...,C_n(t))}{W_2(f_1,...,f_n;g_1,...,g_n;C_1(t),...,C_n(t))}, \tag{5.4f}$$
$$i = 1,...,N, \quad j = 1,\cdots,n.$$

Namely,

$$\alpha\Psi_l[n]_y = \Psi_l[n]_{xx} - 2u[n]\Psi_l[n]_x, \tag{5.5a}$$

$$\alpha\Phi_l[n]_y = -\Phi_l[n]_{xx} - 2u[n]\Phi_l[n]_x, \quad l = 1,...,N+n, \tag{5.5b}$$

$$\alpha\psi_1[n]_y = \psi_1[n]_{xx} - 2u[n]\psi_1[n], \tag{5.5c}$$

$$\psi_1[n]_t = A_1(u[n])\psi_1[n] + T^1_{N+n}(\Psi[n],\Phi[n])\psi_1[n]. \tag{5.5d}$$

So $u[n],\ \Psi_j[n],\ \Phi_j[n],\ j = 1,...,N+n$ *satisfy the mKPESCS (4.6) with degree* $(N+n)$*.*

Proof: By (4.15) and (5.3), we have

$$
\begin{aligned}
\psi_1[n] &= \tfrac{W_3(f_n[n-1],\psi_1[n-1];g_n[n-1];C_n(t))}{W_1(f_n[n-1];g_n[n-1];C_n(t))}\\
&= \tfrac{W_3(f_{n-1}[n-2],f_n[n-2],\psi_1[n-2];g_{n-1}[n-2],g_n[n-2];C_{n-1}(t),C_n(t))}{-C_{n-1}(t)+\int f_{n-1}[n-2]_x g_{n-1}[n-2]\mathrm{d}x}\\
&\quad \times \tfrac{-C_{n-1}(t)+\int f_{n-1}[n-2]_x g_{n-1}[n-2]\mathrm{d}x}{W_1(f_{n-1}[n-2],f_n[n-2];g_{n-1}[n-2],g_n[n-2];C_{n-1}(t),C_n(t))}\\
&= \cdots\\
&= \tfrac{W_3(f_1,f_2,\ldots,f_n,\psi_1;g_1,\ldots,g_n;C_1(t),\ldots,C_n(t))}{W_1(f_1,\ldots,f_n;g_1,\ldots,g_n;C_1(t),\ldots,C_n(t))}.
\end{aligned}
$$

Similarly we can prove (5.4c) and (5.4d) hold.

$$
\begin{aligned}
u[n] &= u[n-1]+\partial_x \ln\tfrac{W_2(f_n[n-1];g_n[n-1];C_n(t))}{W_1(f_n[n-1];g_n[n-1];C_n(t))}\\
&= u[n-2]+\partial_x \ln\tfrac{\int f_{n-1}[n-2]g_{n-1}[n-2]_x\mathrm{d}x+C_{n-1}(t)}{\int f_{n-1}[n-2]_x g_{n-1}[n-2]\mathrm{d}x-C_{n-1}(t)}+\partial_x \ln\tfrac{W_2(f_n[n-1];g_n[n-1];C_n(t))}{W_1(f_n[n-1];g_n[n-1];C_n(t))}\\
&= u[n-2]+\partial_x \ln\tfrac{W_2(f_{n-1}[n-2],f_n[n-2];g_{n-1}[n-2],g_n[n-2];C_{n-1}(t),C_n(t))}{W_1(f_{n-1}[n-2],f_n[n-2];g_{n-1}[n-2],g_n[n-2];C_{n-1}(t),C_n(t))}\\
&= \cdots\\
&= u+\partial_x \ln\tfrac{W_2(f_1,\ldots,f_n;g_1,\ldots,g_n;C_1(t),\ldots,C_n(t))}{W_1(f_1,\ldots,f_n;g_1,\ldots,g_n;C_1(t),\ldots,C_n(t))}.
\end{aligned}
$$

$$
f_j[j]=f_j[j-1]-\frac{f_j[j-1]\int g_j[j-1]f_j[j-1]_x\mathrm{d}x}{\int g_j[j-1]f_j[j-1]_x\mathrm{d}x-C_j(t)}=-\frac{C_j(t)f_j[j-1]}{\int g_j[j-1]f_j[j-1]_x\mathrm{d}x-C_j(t)},
$$

$$
\Psi_{N+j}[j]=-\frac{1}{2}\frac{\sqrt{\dot{C}_j(t)}f_j[j-1]}{\int f_j[j-1]_x g_j[j-1]\mathrm{d}x-C_j(t)},
$$

so

$$
\Psi_{N+j}[j]=\frac{1}{2}\frac{\sqrt{\dot{C}_j(t)}}{C_j(t)}f_j[j].
$$

So

$$
\begin{aligned}
&\Psi_{N+j}[n]\\
=\;& \tfrac{W_3(f_n[n-1],\Psi_{N+j}[n-1];g_n[n-1];C_n(t))}{W_1(f_n[n-1];g_n[n-1];C_n(t))}\\
=\;& \cdots\\
=\;& \tfrac{W_3(f_{j+1}[j],\cdots,f_n[j],\Psi_{N+j}[j];g_{j+1}[j],\cdots,g_n[j];C_{j+1}(t),\cdots,C_n(t))}{W_1(f_{j+1}[j],\cdots,f_n[j];g_{j+1}[j],\cdots,g_n[j];C_{j+1}(t),\cdots,C_n(t))}\\
=\;& \tfrac{\sqrt{\dot{C}_j(t)}}{2C_j(t)}\tfrac{W_3(f_{j+1}[j],\cdots,f_n[j],f_j[j];g_{j+1}[j],\cdots,g_n[j];C_{j+1}(t),\cdots,C_n(t))}{W_1(f_{j+1}[j],\cdots,f_n[j];g_{j+1}[j],\cdots,g_n[j];C_{j+1}(t),\cdots,C_n(t))}\\
=\;& \cdots\\
=\;& \tfrac{\sqrt{\dot{C}_j(t)}}{2C_j(t)}\tfrac{W_3(f_1,\cdots,f_n,f_j;g_1,\cdots,g_n;C_1(t),\cdots,C_n(t))}{W_1(f_1,\cdots,f_n;g_1,\cdots,g_n;C_1(t),\cdots,C_n(t))}\\
=\;& -\tfrac{1}{2}\sqrt{\dot{C}_j(t)}\tfrac{W_3(f_1,\ldots,f_{j-1},f_{j+1},\ldots,f_n,f_j;g_1,\ldots,g_{j-1},g_{j+1},\ldots,g_n;C_1(t),\ldots,C_{j-1}(t),C_{j+1}(t),\ldots,C_n(t))}{W_1(f_1,\ldots,f_n;g_1,\ldots,g_n;C_1(t),\ldots,C_n(t))}.
\end{aligned}
$$

Similarly we can prove (5.4f) holds.
This completes the proof.

Remark: If $C_j(t), j = 1, ..., n$ are not constants, i.e. $\frac{d}{dt}C_j(t) \neq 0$, the DT (5.4) provides a non-auto-Bäcklund transformation between two mKPESCSs (4.6) with degree N and $N+n$ respectively.

6 Some examples of solutions for the mKPESCS

1. **Rational solution.**

Example 1: Rational solution with singularities for the mKPIIESCS ($\alpha = 1$).
If we set $\alpha = 1$ in equation (4.6), we get the mKPIIESCS

$$u_t + u_{xxx} + 3D^{-1}(u_{yy}) - 6D^{-1}(u_y)u_x - 6u^2u_x + 4\sum_{i=1}^{N}(\Psi_i\Phi_i)_x = 0, \tag{6.1a}$$

$$\Psi_{i,y} = \Psi_{i,xx} - 2u\Psi_{i,x}, \tag{6.1b}$$

$$\Phi_{i,y} = -\Phi_{i,xx} - 2u\Phi_{i,x}. \tag{6.1c}$$

We take $u = 0$, $\Phi_1 = ae^{kx-k^2y}$, $\Psi_1 = be^{-kx+k^2y}$, $k, a, b \in \mathbb{R}$ as the initial solution of (6.1) with $N = 1$ and let

$$f_1 = (2x + 8ky - 96k^2t - \frac{8abt}{9k})e^{2kx+4k^2y-32k^3t-\frac{8ab}{3}t}, \quad g_1 = e^{2kx-4k^2y-32k^3t+8abt}, \quad C(t) = 0,$$

then by DT (4.15), we get the rational solution with singularities for the mKPIIESCS (6.1) with $N = 1$ as follows

$$u[1] = \partial_x \ln\frac{\int f_1 g_{1,x}\mathrm{d}x}{\int g_1 f_{1,x}\mathrm{d}x} = \frac{8k}{(2kA+1)(2kA-1)}, \tag{6.2a}$$

$$\Psi_1[1] = 3be^{-kx+k^2y}\frac{A+\frac{1}{6k}}{A+\frac{1}{2k}}, \tag{6.2b}$$

$$\Phi_1[1] = \frac{1}{3}ae^{kx-k^2y}\frac{A-\frac{1}{6k}}{A-\frac{1}{2k}}, \tag{6.2c}$$

where $A = 2x + 8ky - 96k^2t - \frac{8abt}{9k}$.
More generally, if we take

$$f_i = (x + 2k_iy - 12k_i^2t + \frac{4abt}{(k_i+k)^2})e^{k_ix+k_i^2y-4k_i^3-\frac{4abt}{k_i+k}}, \tag{6.3a}$$

$$g_i = e^{k_i x - k_i^2 y - 4k_i^3 + \frac{4k_i abt}{k_i - k}}, \tag{6.3b}$$

$$C_i(t) = 0, \quad k_i \neq \pm k, i = 1, ..., n, k_i + k_j \neq 0, \ \forall i, j, \tag{6.3c}$$

then (5.4b), (5.4c) and (5.4d) will give the rational solution with multi-singularities for the mKPIIESCS with $N = 1$.

Example 2: Lump solution for the mKPIESCS ($\alpha = i$).
If we set $\alpha = i$ in equation (4.6), we get the mKPIESCS

$$u_t + u_{xxx} - 3D^{-1}(u_{yy}) - 6iD^{-1}(u_y)u_x - 6u^2 u_x + 4\sum_{i=1}^{N}(\Psi_i \Phi_i)_x = 0, \tag{6.4a}$$

$$i\Psi_{i,y} = \Psi_{i,xx} - 2u\Psi_{i,x}, \tag{6.4b}$$

$$i\Phi_{i,y} = -\Phi_{i,xx} - 2u\Phi_{i,x}. \tag{6.4c}$$

We take $u = 0$, $\Phi_1 = ae^{-ikx - ik^2 y}$, $\Psi_1 = be^{ikx + ik^2 y}$, $k, a, b \in \mathbb{R}$ as the initial solution of (6.4) with $N = 1$ and let

$$f_1 = (x - 2ly + 12l^2 t - \frac{4abkti}{(k+l)^2})e^{-ilx + il^2 y - 4il^3 t - \frac{4ablt}{k+l}}, \quad g_1 = e^{-ilx - il^2 y - 4il^3 t + \frac{4ablt}{l-k}},$$

$l \in \mathbb{R}$, $l \neq \pm k$ and $C(t) = 0$, then by DT (4.15), we get the 1-lump solution for the mKPIESCS (6.4) with $N = 1$ as follows

$$u[1] = \partial_x \ln \frac{\int f_1 g_{1,x} \mathrm{d}x}{\int g_1 f_{1,x} \mathrm{d}x} = \frac{4li}{1 + (2lA)^2}, \tag{6.5a}$$

$$\Psi_1[1] = be^{kxi + k^2 yi} \frac{-2Al(k+l) + i(k-l)}{2lA(k-l) + i(k-l)}, \tag{6.5b}$$

$$\Phi_1[1] = ae^{-kxi - k^2 yi} \frac{2lA(k^2 - l^2) + (k-l)^2 i}{-2lA(l+k)^2 + i(k+l)^2}, \tag{6.5c}$$

where $A = x - 2ly + 12l^2 t - 4ab\frac{ikt}{(k+l)^2}$.
More generally, if we take

$$f_j = (x - 2l_j y + 12l_j^2 t + \frac{4l_j abt}{(l_j + k)^2})e^{-il_j x + il_j^2 y - 4il_j^3 t - \frac{4il_j abt}{l_j + k}}, \tag{6.6a}$$

$$g_j = e^{-il_j x - il_j^2 y - 4il_j^3 t + \frac{4il_j abt}{l_j - k}}, \tag{6.6b}$$

$$C_j(t) = 0, \quad l_j \neq \pm k, j = 1, ..., n, l_m + l_j \neq 0, \ \forall m, j, \tag{6.6c}$$

then (5.4b), (5.4c) and (5.4d) will give the multi-lump solution for the mKPIESCS with $N=1$.

2. **Soliton solution.**

Example 3: Soliton solution for the mKPIIESCS.

We take $u=0$ as the initial solution for the mKPIIESCS (6.1) with $N=0$ and let

$$f_1=e^{kx+k^2y-4k^3t}=e^{\xi_1},\quad g_1=e^{lx-l^2y-4l^3t}=e^{\xi_2},\quad C(t)=e^{2\beta(t)},$$

where$k,l\in\mathbb{R}$, $k+l\neq 0$, and $\beta(t)$ is an arbitrary function in t. Then by DT (4.15), we get the 1-soliton solution for the mKPIIESCS (6.1) with $N=1$ as follows

$$u[1]=\partial_x\ln\frac{\int f_1g_{1,x}\mathrm{d}x+C(t)}{\int g_1f_{1,x}\mathrm{d}x-C(t)}=-\frac{k+l}{(\frac{l}{k+l}e^{\eta}-e^{-\eta})(\frac{k}{k+l}e^{\eta}+e^{-\eta})},\quad \eta=\frac{\xi_1+\xi_2}{2}-\beta(t),\quad (6.7a)$$

$$\Psi_1[1]=-\frac{1}{2}f_1\frac{\sqrt{\dot{C}(t)}}{\int f_{1,x}g_1\mathrm{d}x-C(t)}=-\frac{\sqrt{2\dot{\beta}(t)}}{2}\frac{e^{\xi_1+\beta(t)}}{\frac{k}{k+l}e^{\xi_1+\xi_2}-e^{2\beta(t)}},\quad (6.7b)$$

$$\Phi_1[1]=\frac{1}{2}g_1\frac{\sqrt{\dot{C}(t)}}{\int g_{1,x}f_1\mathrm{d}x+C(t)}=\frac{\sqrt{2\dot{\beta}(t)}}{2}\frac{e^{\xi_2+\beta(t)}}{\frac{l}{k+l}e^{\xi_1+\xi_2}+e^{2\beta(t)}}.\quad (6.7c)$$

More generally, if we take

$$f_i=e^{k_ix+k_i^2y-4k_i^3t},\quad g_i=e^{l_ix-l_i^2y-4l_i^3t},\quad C_i(t)=e^{\beta_i(t)},i=1,...,n,\quad (6.8)$$

where $k_i,l_i\in\mathbb{R}$, $k_i+l_j\neq 0$, $\forall i,j$, then (5.4b), (5.4e) and (5.4f) will give the n-soliton solution for the mKPIIESCS with $N=n$.

Example 4: Soliton solution for the mKPIESCS.

We take $u=0$ as the initial solution for the mKPIESCS (6.4) with $N=0$ and let

$$f_1=e^{-ikx+ik^2y-4ik^3t},\quad g_1=e^{i\bar{k}x-i\bar{k}^2y+4i\bar{k}^3t},\quad C(t)=ie^{2\beta(t)}$$

where $k\in\mathbb{C}$ and $\beta(t)$ is an arbitrary function in t.

Set $\quad k=\mu-i\nu,\quad \mu,\nu\in\mathbb{R},\quad \nu\neq 0,$

then

$$f_1=e^{\theta+\eta},\quad g_1=\bar{f}_1=e^{-\theta+\eta}$$

where

$$\theta=-i\mu x+i(\mu^2-\nu^2)y-4i(\mu^3-3\mu\nu^2)t,\quad \eta=-\nu x+2\mu\nu y+4\nu(\nu^2-3\mu^2)t.$$

Then by DT (4.15), we get the 1-soliton solution for the mKPIESCS (6.4) with $N = 1$ as follows

$$u[1] = \partial_x \ln \frac{\int f_1 g_{1,x} \mathrm{d}x + C(t)}{\int g_1 f_{1,x} \mathrm{d}x - C(t)} = \frac{2\nu i}{\frac{1}{4}e^{2f} + (e^{-f} - \frac{\mu}{2\nu}e^f)^2}, \quad f = \eta - \beta(t) \tag{6.9a}$$

$$\Psi_1[1] = -\frac{1}{2} f_1 \frac{\sqrt{\dot{C}(t)}}{\int f_{1,x} g_1 \mathrm{d}x - C(t)} = \frac{\sqrt{\dot{\beta}(t)}(1-i)e^{\theta + \nu x + 2\mu\nu y + 4\nu^3 t + 12\mu^2 \nu t}}{(i\nu - \mu)e^{8\nu^3 t + 4\mu\nu y} + 2\nu e^{\beta(t) + 24\mu^2 \nu t + 2\nu x}}, \tag{6.9b}$$

$$\Phi_1[1] = \frac{1}{2} g_1 \frac{\sqrt{\dot{C}(t)}}{\int g_{1,x} f_1 \mathrm{d}x + C(t)} = \frac{\sqrt{\dot{\beta}(t)}(i-1)e^{-\theta + \nu x + 2\mu\nu y + 4\nu^3 t + 12\mu^2 \nu t}}{(i\nu + \mu)e^{8\nu^3 t + 4\mu\nu y} - 2\nu e^{\beta(t) + 24\mu^2 \nu t + 2\nu x}}, \tag{6.9c}$$

More generally, if we take

$$f_j = e^{-ik_j x + ik_j^2 y - 4ik_j^3 t}, \quad g_j = e^{i\bar{k}_j x - i\bar{k}_j^2 y + 4i\bar{k}_j^3 t}, \quad C_j(t) = ie^{2\beta_j(t)}, j = 1, ..., n, \tag{6.10}$$

where $k_j = \mu_j + i\nu_j$, $\mu_j, \nu_j \in \mathbb{R}$, $k_j \neq \bar{k}_m$, $\forall j, m$ and $\beta_j(t)$, $j = 1, ..., n$, are arbitrary functions in t, then (5.4b), (5.4e) and (5.4f) will give the n-soliton solution for the mKPIESCS with $N = n$.

3. **Solutions of breather type.**

Example 5: Solutions of breather type for the mKPIESCS.

We take $u = 0$ as the initial solution for the mKPIESCS (6.4) with $N = 0$. If we take

$$f_j = e^{-i\lambda_j x + i\lambda_j^2 y - 4i\lambda_j^3 t}, \quad g_j = e^{i\xi_j x - i\xi_j^2 y + 4i\xi_j^3 t}, \quad C_j(t) = ie^{2\beta_j(t)}, \quad j = 1, ..., 2n$$

where

$$(\lambda_1, ..., \lambda_{2n}) = (k_1, ..., k_n; l_1, ..., l_n), \qquad (\xi_1, ..., \xi_{2n}) = (\bar{l}_1, ..., \bar{l}_n; \bar{k}_1, ..., \bar{k}_n),$$

$k_j, l_j \in \mathbb{C}$, $Im(k_j) \neq 0$, $Im(l_j) \neq 0$, $l_m \neq \bar{k}_j$, $\forall m, j$,
we will get the solutions of breather type for the mKPIESCS by (5.4b), (5.4e) and (5.4f). For example, if we choose $n = 1$ and

$$\lambda_1 = k_1 = -bi, \quad \lambda_2 = l_1 = -di, \quad \xi_1 = \bar{\lambda}_2 = di, \quad \xi_2 = \bar{\lambda}_1 = bi, \quad C_1(t) = C_2(t) = ie^{2t},$$

we will get the following solution of mKPIESCS

$$u[2] = -\frac{8(b+d)^2 \cos\theta \mathrm{e}^{\mathrm{f}} \mathrm{i}}{4(b+d)^2 e^{2t} + 4(b^2 - d^2)e^f + (b-d)^2 e^{2f-2t}}, \tag{6.11a}$$

$$\Psi_1[2] = \frac{1+i}{2} e^{\eta_1 + \theta_2 i + t} \frac{\frac{d-b}{2(d+b)} e^f + e^{2t}\sin\theta - \mathrm{i}\mathrm{e}^{2\mathrm{t}}\cos\theta}{e^{4t} - e^{f+2t}\sin\theta \frac{\mathrm{b-d}}{\mathrm{b+d}} + \frac{(\mathrm{b-d})^2}{4(\mathrm{b+d})^2}\mathrm{e}^{2\mathrm{t}} + \mathrm{e}^{\mathrm{f+2t}}\cos\theta \mathrm{i}}, \tag{6.11b}$$

$$\Psi_2[2] = \frac{1+i}{2} e^{\eta_2+\theta_1 i+t} \frac{\frac{b-d}{2(d+b)} e^f - e^{2t}\sin\theta - \mathrm{i}e^{2t}\cos\theta}{e^{4t} - e^{f+2t}\sin\theta \frac{b-d}{b+d} + \frac{(b-d)^2}{4(b+d)^2} e^{2t} + e^{f+2t}\cos\theta \mathrm{i}}, \tag{6.11c}$$

$$\Phi_1[2] = -\frac{1+i}{2} e^{\eta_2-\theta_1 i+t} \frac{\frac{b+d}{2(b-d)} e^f - e^{2t}\sin\theta + \mathrm{i}e^{2t}\cos\theta}{e^{4t} - e^{f+2t}\sin\theta \frac{b-d}{b+d} + \frac{(b-d)^2}{4(b+d)^2} e^{2t} - e^{f+2t}\cos\theta \mathrm{i}}, \tag{6.11d}$$

$$\Phi_2[2] = -\frac{1+i}{2} e^{\eta_1-\theta_2 i+t} \frac{\frac{b+d}{2(d-b)} e^f + e^{2t}\sin\theta + \mathrm{i}e^{2t}\cos\theta}{e^{4t} - e^{f+2t}\sin\theta \frac{b-d}{b+d} + \frac{(b-d)^2}{4(b+d)^2} e^{2t} - e^{f+2t}\cos\theta \mathrm{i}}, \tag{6.11e}$$

where

$$f = -(b+d)x + 4(b^3+d^3)t, \quad \theta = (d^2-b^2)y, \quad \eta_1 = -bx + 4b^3 t, \quad \eta_2 = -dx + 4d^3 t,$$

$$\theta_1 = -b^2 y, \quad \theta_2 = -d^2 y.$$

$u[2]$ is periodic in y and has soliton behavior along the coordinate x.
Similarly, we can get the solution of breather type for the mKPIIESCS.

Remark: In **Example 1** and **Example 2**, when $a = b = 0$, the solutions obtained above will degenerate to the solutions of the corresponding mKP equations respectively. In **Example 3**, **Example 4** and **Example 5**, when $C(t)(C_j(t))$ are taken to be constant(s), i.e. $\frac{d}{dt}C(t)(\frac{d}{dt}C_j(t)) = 0$, the solutions obtained above will also degenerate to the solutions of the corresponding mKP equations respectively [24].

Acknowledgment

This work was supported by the Chinese Basic Research Project "Nonlinear Science".

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